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HYPERSTABLE MODEL REFERENCE ADAPTIVE CONTROL FOR A CLASS OF NONLINEAR PLANTS

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by

Roberto Jacobo Benhabib

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HYPERSTABLE MODEL REFERENCE ADAPTIVE CONTROL FOR A CLASS OF NONLINEAR PLANTS

BY

ROBERTO JACOBO BENHABIB

B.S., University of Illinois, 1976

THESIS

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CHAPTER 1

INTRODUCTION

In this study we will be concerned with the development of a hyperstable model reference adaptive control for a class of nonlinear plants. These results represent an extension of the Landau [19] method for the linear time invariant plant to the nonlinear case.

In recent years there has been much engineering interest in adaptive control systems. One of the reasons for this interest is that modeling of systems often introduces parameter value and structure errors. When these errors are carried through to the control design stage, they often produce unacceptable controls for the real system. Hence, a controller which "adapts" to these errors so as to stabilize the dynamic characteristics of the system about the specified response would be highly advantageous. This type of control was believed to have been first developed by Whitaker, Yamron, and Kezer [1] in 1958. Since then, their initial work has motivated an adaptive control theory encompassing many different adaptation approaches. It has been the success of these techniques to problems ranging from missile flight control to biological systems to economics and source management [2], that motivate this study.

As is suggested above, the adaptive control problem often arises when exact modeling is infeasible. However, if the main source of error is of the structure type, little can presently be done except for some special cases. Hence, the discussion is limited to parameter adaptation. The following background information is presented to expose the underlying problem and mathematical treatment involved.

The parameter adaptation problem has many sources. As an example, consider the following typical control problem: Given a plant with some inputs and outputs, design a controller such that the plant responds in a prespecified manner. To solve this problem, one would probably first try modeling the plant process with some mathematical structure, such as a set of differential equations, which together with some uniquely defining parameters, would reasonably describe the physical behavior of the plant. One would then search for a controller for this model and then hope that the same controller would work in the plant. However, this is often an infeasible solution for the following reason: Parameter errors will exist whenever the modeling fails to take large parameter variations into account. These parameter variations often take the physical form of parts wearing out, manufacturing tolerances in the machine parts, or as physical variations due to the different operating conditions of the equipment (as is the case of the space shuttle). The net effect in some of these cases may be unacceptable performance or instability in the plant. In fact, under some circumstances the initial parameter values may be indeterminable and time varying so the conventional design may be infeasible. Thus parameter errors are inherent in some control applications. The question is whether they can be neglected in a particular case, and if not, what can be done about it.

In some cases, the solution lies in parameter adaptive control, which for this study, will be defined as in Landau [3]:

"An adaptive system measures a certain index of performance using the inputs, the states, and the outputs of the adjustable system. From the comparison of the measured index of performance values and a set of given ones,

the adaptation mechanism modifies the parameters of the adjustable system or generates an auxiliary input in order to maintain the index of performance values close to the set of given ones."

Thus, if such an adaptive controller could be realized, one would be able to solve the above control problem despite the existence of parameter value uncertainty. The case with unknown parameters, though actually more difficult to treat mathematically, would similarly be solved.

The methodologies required to analyze and synthesize the adaptive controller falls into one of two categories: the stochastic methods and the deterministic methods. Stochastic methods are those methods which introduce random models of uncertainty and assign probability measures to the unknown parameters. An example of this type is dual control [31]. In this approach, a system is simultaneously identified and controlled so that as the parameters are estimated, a control is implemented in such a way, that for these estimates, the system behaves in some ideal sense. As the estimates are made to converge in a probabilistic sense, the control will be forced to converge to those values that elicit the ideal behavior in the system. Deterministic methods, on the other hand, use mathematical concepts such as stability and algorithmic minimization to force adaptation. An example of this type is model reference adaptive control (M.R.A.C.), which will now be discussed.

CHAPTER 2

THE M.R.A.C. METHOD AND THE STABILITY APPROACH

In the Introduction, it was suggested that many control problems subject to parameter perturbations may be solved using a M.R.A.C. But what is a M.R.A.C. and how is it realized? To answer these questions we now review some standard definitions and results which may be useful in understanding the underlying concepts of M.R.A.C. and also make clearer the difficulties of the methods involved [3].

The M.R.A.C. system may be defined as an adaptive control system which uses an external model as a reference for measuring the performance index of the plant (see Figure 1). It will be shown that this method may be employed to force the plant responses to follow the responses of the reference model.

In the system structure of Figure 1, the reference model may be real or simulated. The error vector, representing the deviation between the real and ideal responses, is defined as the difference between the plant and model outputs. Generally, this vector may include any or all of the state variables along with other outputs. Once the "performance measures" are taken, the errors are then fed back through an adaptive algorithm which will either generate parameter perturbations or alter the input command u, depending, respectively, upon whether the parameters are physically available for change or not. It will be this adaptive loop which forces the error vector and its derivative to approach zero in the case of perfect model reference adaptation.

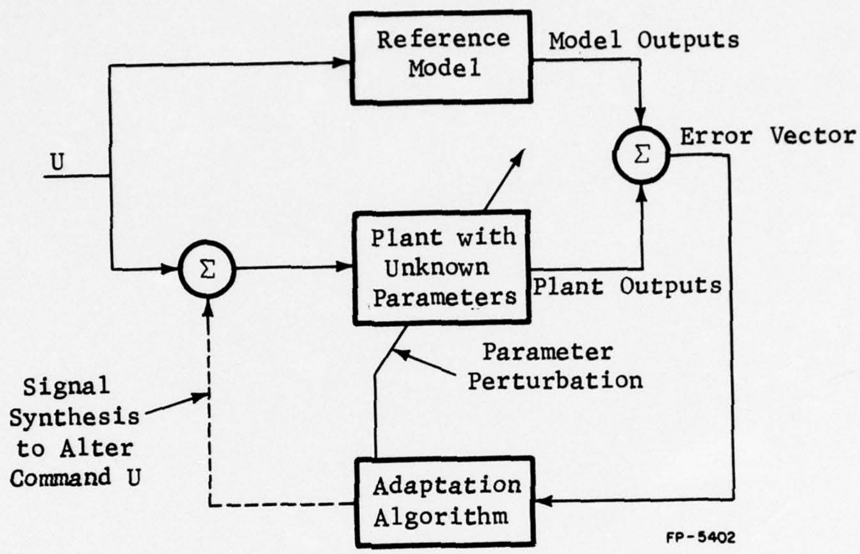

Figure 1. M.R.A.C. Structure

originating with Rucker [14], state-variable-filters (SVF) exploited the commutation of linear operators in linear systems and thus were able to successfully measure equivalent state variables without differentiation. This was indeed a very important step since differentiation, the alternative to SVF, has adverse affects on noisy measurements. The SVF concept was then coupled with the Kalman-Yakubovich Lemma [15] by Monopoli [16], Narendra [10], and others to complete the solution of the L.T.I. M.R.A.C. in 1972-3. The additional effects of the Lemma were to guarantee a solution for the L.T.I. problem when the linear reference model was asymptotically stable and also reduce the error vector stability equation to a set of scalar equations defining the adaptation algorithm.

The hyperstability approach originated in 1969 when Landau [17] first proposed it as an alternative for the solution of the L.T.I. M.R.A.C. problem. Landau later generalized the class of potential adaptive laws [18] and then with the aid of SVF almost completed the L.T.I. case design [19]. A general solution for generating a positive real function given totally unknown parameters is as yet unknown.

The M.R.A.C. for nonlinear plants represents an additional level of complexity, hence the solution has developed more slowly. One of the earlier contributions was published by Lindorff and Monopoli [20] in 1967. Using Lagrange stability theorems, they produced a solution requiring restrictions on the input-output relations, knowledge of the parameter and parameter derivative bounds, and all state variable measurement. The solution was then broadened by Lal and Mehrotra [21] in 1972 using an absolute stability theorem by Davison [22]. This solution required state variable measurements and

restricted the formulation for the reference model, but it allowed a very general calss of nonlinearities in the plant and no parameter bound requirements. Lowe and Rowland [23] then introduced a Lyapunov stable M.R.A.C. which required parameter and parameter derivative bounds, but allowed very general nonlinearities and simplified the adaptive control to the bang-bang type. Finally, the most recent major improvement came from Monopoli and Hsing [24] in 1975 when they introduced the SVF concept into the nonlinear problem. This was a welcome result since it reduced the need for differentiation. However, to apply the filters, they had to make several restrictions as discussed below.

It was shown by Monopoli and Hsing [24] that SVF may be used to reduce the need for all-state-variable measurement in a certain class of plants. This class of plants is of the following form:

$$D_i(p)\, x_i(t) = \sum_{j=1}^{\ell} N_{ij}(p)\, u_j(t) + c_i f_i(x_1,\ldots x_q, t)\ , \qquad i=1,\ldots,q$$

where $u_j(t)$, $j=1$ to ℓ and x_i, $i=1$ to q are the plant inputs and outputs respectively; $f_i(x_1,\ldots,x_q,t)$ are nonlinear time-varying functions of known form; p is the differential operator d/dt;

$$D_i(p) = p^{m_i} + a_{i1}p^{m_i-1} + \ldots + a_{im_j}$$

$$N_{ij}(p) = b_{ij}p^{m_{ij}} + b_{ij_1}p^{m_{ij}-1} + \ldots + b_{ijm_{ij}}\ , \qquad i=1,\ldots,q$$

a_{ij}, b_{ij_k} and c_i are unknown constant or slow-time-varying parameters; plus other assumptions.

Clearly, under the above conditions, all the state variables are functions involving linear operations on the outputs, and the f_i's are not

functions of derivatives in x_q. Therefore linear operators (i.e., SVF) may be used to recover the states from the outputs without derivatives. However, when the above assumptions are violated, SVF are of little use since linearity can no longer be exploited. Therefore, due to lack of a better technique, plant states must generally be measured.

The above development shows that several stability synthesis techniques have been used to treat the nonlinear case. The Lyapunov and hyperstability methods, however, appeared to be the most general. Hence, we now review some standard definitions and theorems with the intention of outlining and comparing the two methods.

Definition [5]: A function $V(x)$, $V:R^m \to R$, $x \in R^m$ is a Lyapunov function if it satisfies the following conditions:

1) $V(x)$ is continuous and has continuous first partial derivatives in some open region R about the origin, defined by $\|x\| < a$, where a is a positive constant.
2) $V(x)$ is positive definite in the region R.
3) Relative to the system $\dot{x}=F(x)$, F a function vector, the time derivative $\dot{V}(x)$ along the trajectories of the system is negative semidefinite in R.

Theorem (Lyapunov) [5]: If for the system $\dot{x}=F(x)$, $F(0)=0$, a Lyapunov function can be found in some neighborhood R about the origin, then the origin is stable.

Theorem (Lyapunov) [5]: For the system $\dot{x}=F(x)$, $F(0)=0$. If in a neighborhood R about the origin a Lyapunov function V(x) can be found such that $\dot{V}$ is negative definite, then the origin is asymptotically stable.

The following will be used to define hyperstability [6]: Consider the system described by the state equations of the form

$$\dot{x} = F\,x + G\,u$$
$$y = H\,x + J\,u \tag{1}$$

where the triple (F,G,H) is observable and controllable, and u and y have the same dimension. Define $u(\cdot)$ such that for all T

$$\int_{o}^{T} u'(t)y(t)dt \leq \delta[\|x(0)\|] \sup_{o \leq t \leq T} \|x(t)\| \tag{2}$$

where δ is a positive constant depending on the initial state x(0) of the system but independent of the time T and the norm is the Euclidean norm.

Definition (Popov) [6]: The system (1) is termed hyperstable if for any $u(\cdot)$ in the subset defined by (2), the following inequality holds for some positive constant K and all t:

$$\|x(t)\| \leq k(\|x(o)\| + \delta) \tag{3}$$

If in addition $u(\cdot)$ is bounded and $\lim_{t\to\infty} x(t) = 0$, the system is termed asymptotically hyperstable.

Definition [7]: A rational fractional function M(s) of the complex variable s, real valued for real values s is called positive real if

1) Re $M(jw) \geq 0 \quad \forall W \in \mathbb{R}$,

2) it has no right-hand-plane poles, and

3) it has single imaginary poles with real and positive residues.

Definition [6]: A rational transfer-function matrix $Z(s)$ is termed positive real if the following three conditions are satisfied:

1) $Z(s)$ has real elements for real s.

2) The elements of $Z(s)$ are positive real.

3) For any real value of W such that no element of $Z(jw)$ has a pole for this value, $z(jw) + z'^{*}(jw)$ is positive semidefinite Hermitian.

Theorem (Popov) [6]: The system defined by (1) and (2) is hyperstable, if and only if the transfer function of (1) is positive real.

Theorem (Popov) [6]: The system defined by (1) and inputs (2) is asymptotically hyperstable, if and only if the transfer function $z(s) = H(SI-F)^{-1}G+J$ for the system (1) is strictly positive real.

Using the Lyapunov theorems and the Lyapunov functions together with some algebra, it was shown by Parks [8], Monopoli [9], and Narendra [10] that the M.R.A.C. problem reduces to synthesizing a Lyapunov function such that if the error vector of Fig. 1 is chosen as the state, the associated error system is asymptotically stable. This synthesis generally involves the introduction of a Lyapunov function V which is quadratic in the error vector, forming the associated $\dot{V}$ and then manipulating parameter perturbation signals in $\dot{V}$ such that $\dot{V}$ is negative definite. Thus, the generation of the

V function and the manipulation of the parameter signal perturbations in $\dot{V}$ are the key to the solution using the Lyapunov approach, both of which are often difficult in general.

In hyperstability synthesis, on the other hand, one begins by restructuring the system of Fig. 1 so that the error vector is defined as the state and a partioning of forward and feedback system signals yields a linear forward path. One then designs a constant matrix D such that, if D operates on the forward path signals, the overall forward transfer function is positive real (see Fig. 3)*. The synthesis is completed by finding functionals for the forward signals such that, when fed back through the feedback path, the input-output relations of the feedback path satisfy an inequality analogous to (2). Hence, the M.R.A.C. problem is reduced to a three step process, the last two steps being difficult to do.

Since Lyapunov and hyperstability synthesis are both difficult to apply and have equally desiable stability properties, the choice depends upon the particular application. In this paper the hyperstability was chosen since few results existed for the hyperstable nonlinear M.R.A.C. and more importantly because it appeared to be the simpler design.

---

* The new structure is called the Equivalent Popov Structure.

TABLE 1

SUMMARY OF LYAPUNOV AND HYPERSTABILITY

DESIGN

LYAPUNOV	HYPERSTABILITY METHOD USED HERE
1. Form error vector between model and plant states.	1. Same as Lyapunov
2. Define the error vector as the state of the system and algebraically manipulate plant and model state equations to form system state equations of the form $\dot{e} = f(e, x_m, x_p, u, t)$	2. Same as Lyapunov
3. Find a Lyapunov function and form $\dot{V}$.	3. Choose μ as the sum of variable gains times the functions of x_m and x_p appearing in the plant and model.
4. Try to define u as a function of x_m, x_p and t such that $\dot{V}$ is negative definite. If such an u exists, design is finished. If not, go back to step 3.	4. Substitute u from step 3 into f of step 2 and separate f into linear and nonlinear parts.
	5. Define a matrix D such that D times the transfer function of the linear part found in step 4 is positive real.
	6. Try to find variable gains of step 3 as a function of x_m, x_p such that input-output relations of the nonlinear part found in step 4 satisfy an integral inequality analog to (2). If this step and step 5 are satisfied, solution is complete.

CHAPTER 3

SOLUTION FOR THE NONLINEAR M.R.A.C. PROBLEM USING HYPERSTABILITY

In this chapter it will be shown that the design method of multivariable adaptive model following control described by Landau [19] may be extended to include a class of nonlinear multivariable plants.

<u>ASSUMPTIONS</u>:

1. The nonlinear plants to be treated here are of the form:

$$\dot{x}_p = A_p x_p + \sum_{\alpha=1}^{\beta} R_p^{\alpha} F_p^{\alpha}(x_p) + B_p u_p \qquad (4)$$

where x_p is an m-dimensional state column vector available from noiseless measurements; u_p is an r-dimensional input column vector; A_p and R_p^{α} for $\alpha=1,\ldots,\beta$ are mxm unknown constant matrices; B_p is an mxr constant matrix whose possibly unknown elements b_{ij} which must lie within bounds defined by the real numbers p_{ij} and q_{ij}; $F_m^{\alpha}(x_p)$ for $\alpha=1,\ldots,\beta$ is an n-dimensional column vector of known nonlinear functions of the state x_p; the plant (4) satisfies the existence and uniqueness properties for a solution (i.e., Global Lipschitz conditions [33]); and β = greatest integer of [k/m], where k is defined as the minimum number of functions f_i needed to represent each of the plant nonlinearities F_j in the form $F_j = \sum_{i=1}^{k} a_{ij} f_i$ with a_{ij}'s elements of the real numbers.

An example for the last condition is:

$$\dot{x}_1 = a_{11}x_1 + a_{13}x_3 + r(x_2x_3 + \cos x_1) + b_{11}u_1$$

$$\dot{x}_2 = a_{22}x_2 + r_{22} \cos x_1$$

$$\dot{x}_3 = a_{33}x_3 + b_{32}u_2$$

Here β = greatest integer of [2/3], implies β=1 and

$$\dot{x}_p = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} x_p + \begin{bmatrix} r & r & 0 \\ 0 & r_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2x_3 \\ \cos x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} b_{11} & 0 \\ 0 & 0 \\ 0 & b_{32} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Essentially we are considering nonlinearities f which factor with respect to a set of parameters. That is, for parameter vector P, $f(x,P) = PF(x)$ where F is a column vector of appropriate dimension.

2. The model structure will be given by

$$\dot{x}_m = A_m x_m + \sum_{\gamma=1}^{\rho} R_m^{\gamma} F_m^{\gamma}(x_m) + B_m u_m \qquad (5)$$

where x_m is an m-dimensional state column vector; u_m is an ℓ-dimensional command input column vector; A_m is a known $n \times n$ Hurwitz matrix; B_m and R_m^{γ} for $\gamma = 1,\ldots,\rho$ are $n \times \ell$ and $n \times n$ known constant matrices respectively; $F_m^{\gamma}(x_m)$ for $\gamma = 1,\ldots,\rho$ is an n-dimensional column vector of the nonlinear functions of the states; and ρ is defined similarly to β but with respect to the model nonlinearities.

3. The plant and model satisfy perfect model following conditions.

"Model following is a method of control in which the plant is controlled to behave like an ideal model." ____When it is "possible to match perfectly the dynamics of the compensated plant with those of the model" it is called perfect model following or matching [25]. Hence, perfect model following conditions are those conditions insuring the existence of a feedback compensating control such that the plant and model are input-output equivalent.

Under some circumstances (Chan [25] for the linear case), it can be shown that when perfect model following conditions are not satisfied, the error may still be bounded. Therefore without prior knowledge of the plant, perfect model following may still be useful. Sufficient conditions for perfect model following for the class of nonlinear plants treated here will be derived in Appendix B.

Comment: Due to the distributive nature of the nonlinearities and the technique to be shown, it will only be necessary to show the solution for $\beta=1$ and $\rho=1$. Hence, it will be assumed that $\beta=1$ and $\rho=1$ in all that follows unless stated otherwise. The treatment for $\beta>1$ and/or $\rho>1$ will become obvious as we proceed.

STRUCTURE OF THE CONTROLLER:

There are two basic adaptive model following structures, signal synthesis and parameter adaptation. While the two are mathematically equivalent [19], signal synthesis provides adaptation by modifying the feedback gains instead of directly changing the parameters as is the case in the latter.

Since usually not all the parameters are available for adaptation, signal synthesis is the natural choice. Figure 2 shows how the signal synthesis structure is used in our particular case.

Having chosen signal synthesis, we define

$$
\begin{aligned}
u_p &\triangleq u_p^1 + u_p^2 \\
u_p^1 &\triangleq -K_p x_p + K_m x_m + K_{R_m} F_m + K_{RP} F_p + K_v R_m \\
u_p^2 &\triangleq \Delta K_p x_p + \Delta K_{R_p} F_p + \Delta K_{R_m} F_m + \Delta K_u u_m
\end{aligned}
\tag{6}
$$

where u_p is the plant control signal; u_p^1 is some nominal control whose design shall be discussed later; u_p^2 is the adaptive control; K_p, K_m, K_m, K_{Rp}, ΔK_p, ΔK_m, and K_{R_m} are r×n weighting matrices; and K_u and ΔK_u are also r×ℓ weighting matrices.

There are two reasons for choosing the feedback signal input structure (6). First, recall that the parameters factor out of the known nonlinearities. Hence, it appears that under certain conditions, parameter adaptation may be used to add and subtract functions in the plant state equation by feeding back the required functions and states. Justification for the problem treated appears later and in Appendix B.

The second reason for choosing the structure (6) is that it may be desirable to bias the input control level. For example, if circuit implementation places magnitude constraints upon the adaptive gains and rough estimates are available for the parameters, conventional feedback may be used to drive

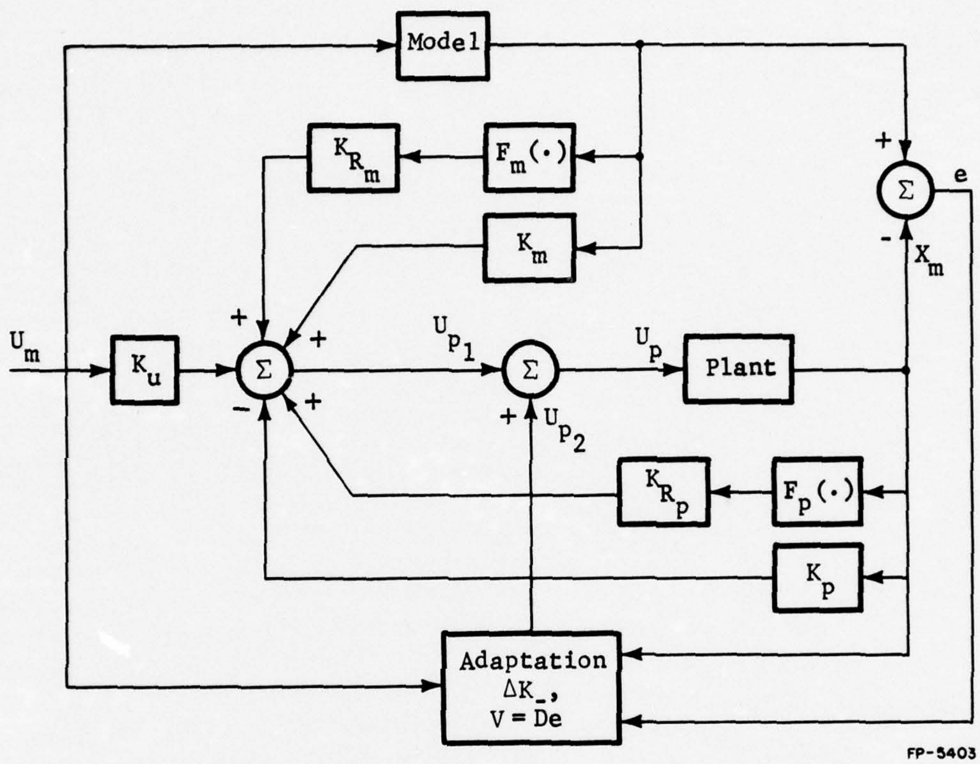

Figure 2. The M.R.A.C. Structure Coupled with a Signal Synthesis Adaptive Control

the system within the range of the adaptive gains. In this problem, one biases the control by defining u_p^1 as a coarse nonadaptive control stabilizing the system for the assumed nominal values of the plant. Likewise u_p^2 is defined as a fine adjustment adapting to any perturbations from the assumed values.

It should be stressed that the assumed nominal values need not be close to the actual values, but the closer they are, the better the biasing effect.

PERFECT MATCHING AND THE EQUIVALENT POPOV STRUCTURE:

Having chosen our structure and controls, one must ask under what conditions will the control force the plant to follow the model. This question is partially answered in Appendix B, where it is shown that sufficient conditions for perfect model following are that the error vector $\underline{e}$ equal zero and that equation (B1) be satisfied. It is also shown that control law (B2) will work under the above conditions.

Prior to using the Assertion of Appendix B to formulate the nominal control u_p^1, we must first assume that (B1) is satisfied. Justifying this point is difficult since it is impossible to tell a priori whether B_p is non-singular or whether the matrices $(A_m - A_p)$, B_m, R_m and R_p lie in the null space of $(I - B_p B_p^+)$. However, given some prior information we may be able to deduce that (B1) is true. If not, we may take comfort in remembering that (B1) is only a sufficient condition. Also Chan [25] has shown that even without perfect following our error may still be bounded and thus the control may still be acceptable.

Accepting (B1) to be true, we choose K_m, K_p, K_u, K_{R_p}, and K_{R_m} by setting the Δ K's = 0 and choosing nominal values for the parameters in (B2). That is,

$$K_m - K_p = \bar{B}_p^+ (A_m - \bar{A}_p) \quad ; \quad K_v = \bar{B}_p^+ B_m ;$$

$$K_{R_p} = -\bar{B}_p^+ \bar{R}_p \quad ; \quad K_{R_m} = \bar{B}_p^+ R_m ;$$

where the "-" means nominal.

Note that the ΔK's will be designed to compensate for any errors in the assumed nominals and setting K_m=0 does not affect the controller.

Now consider the design of the Δ K's. We will construct an equivalent Popov structure and then choose adaptive laws in such a way that will make the overall system asymptotically hyperstable.

The selection of the ΔK's will be accomplished using the hyperstability concepts that were described earlier. The M.R.A.C. structure of Figure 2 will be restructured to an equivalent Popov structure. This restructuring is not difficult to do and forces the M.R.A.C. structure of Figure 2 to be characterized by a system having a linear forward path and nonlinear feedback path in the error system.

The following procedure will produce an equivalent Popov structure for system described by (4), (5), and (6):

Define the error as

$$e \triangleq x_m - x_e \tag{7}$$

Subtracting (4) from (5) and substituting (6) and (7) yields

$$\dot{e} = A_m x_m + R_m F_m + B_m u_m - A_p x_p - R_p F_p - B_p[(\Delta K_p - K_p)x_p$$

$$+(\Delta K_{R_p} + K_{R_p}) F_p + K_m x_m + (K_{R_m} + \Delta K_{R_m}) F_m + (K_u + \Delta K_u)u_m]$$

Now adding and subtracting terms on the right

$$\begin{aligned} \dot{e} = {} & (A_m - B_p K_m) e + B_p(B_p^+ R_m - K_{R_m} - \Delta K_{R_m}) F_m + B_p[B_p^+ B_m - K_u - \Delta K_u] \\ & u_m + B_p[B_p^+(A_m - A_p) + K_p - \Delta K_p - K_m]x_p \\ & - B_p[B_p^+ R_p + K_{R_p} + \Delta K_{R_p}]F_p \end{aligned} \tag{8}$$

Then separate (8) into linear and nonlinear parts by defining

$$\begin{aligned} W \triangleq -W_1 \triangleq {} & (K_{R_m} + \Delta K_{R_m} - B_p^+ R_m)F_m \\ & + (K_u + \Delta K_u - B_p^+ B_m)u_m \\ & + (K_m + \Delta K_p - K_p - B_p^+(A_m - A_p)]x_p \\ & + [B_p^+ R_p + (\Delta K_{R_p} + K_{R_p})]F_p \end{aligned} \tag{9}$$

and substituting (9) into (8) we obtain

$$\dot{e} = (A_m - B_p K_m)e + B_p W_1 \tag{10}$$

Define the following linear operator

$$V = De \tag{11}$$

Therefore equations (9), (10) and (11) establish the equivalent Popov structure for the system (4), (5) and (6). See Figure 3.

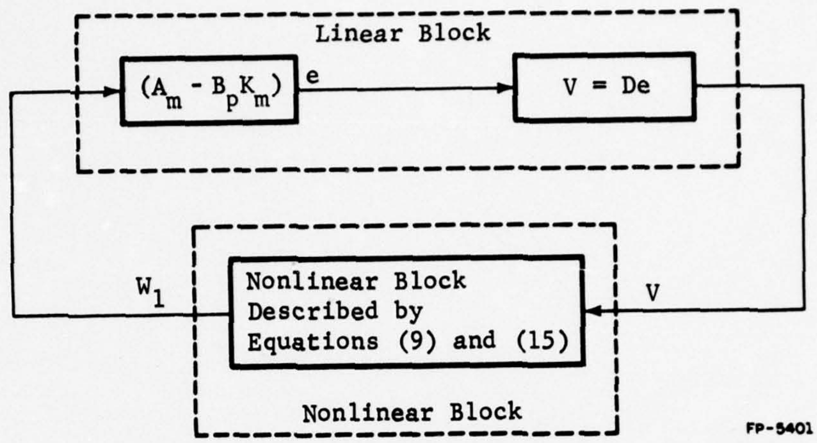

Figure 3. The Equivalent Popov Structure

Using the definition of hyperstability given earlier, we reiterate that the system described by (7), (9), (10), and (11) is hyperstable if there exists constants $\delta \geq 0$ and $y \geq 0$ such that any solution of the system satisfies the inequality

$$\|e(t)\| \leq \delta(\|e(0)\|) + y \ ; \quad \forall t \geq 0 \ ; \quad \delta > 0 \ ; \quad y \geq 0 \tag{12}$$

for all w and v such that

$$\eta(0,t) = \int_o^t W^T V \, d\tau \geq -y_o^2 - y_o \ ; \quad \sup_{o \leq y \leq t} \|e(\tau)\| y_o \geq 0 \tag{13}$$

and is asymptotically hyperstable if in addition to the above $\lim_{t \to \infty} e(t) = 0$ for all W and V satisfying (13).

Therefore, by the Popov theorem necessary and sufficient conditions that the system described by (7), (9), (10), and (11) be asymptotically hyperstable is that the transfer function matrix characterizing the linear part of the equivalent Popov structure,

$$Z(S) = D[SI - (A_m - B_p K_m)]^{-1} B_p \ , \tag{14}$$

be strictly positive real.

Thus we can force $e \to o$ by searching for a constant matrix D to make Z(s) strictly positive real and by choosing the ΔK's in (9) in such a way that (13) is satisfied.

<u>THE ADAPTATION RULES AND THE MATRIX D</u>:

The adaptation rules are the adaptation algorithms that will guarantee the error system is asymptotically hyperstable, and the following assertion provides such an algorithm.

Assertion: The following choices for the ΔK's satisfy (13):

$$
\begin{aligned}
\Delta K_{R_m} &= \int_o^t \tilde{L}V[SF_m]^T \, d\tau + LV(SF_m]^T + \Delta K_{R_m}(0) \\
\Delta K_{R_p} &= \int_o^t \tilde{O}V[QF_p]^T \, d\tau + OV[QF_p]^T + \Delta K_{R_p}(0) \\
\Delta K_p &= \int_o^t \tilde{N}V[PX_p]^T \, d\tau + NV[PX_p]^T + \Delta K_p(0) \\
\Delta K_u &= \int_o^t \tilde{M}V[Tu_m]^T \, d\tau + MV[Tu_m]^T + \Delta K_u(0)
\end{aligned}
\tag{15}
$$

where $\tilde{L}, L, S, \tilde{M}, M, T, \tilde{N}, N, P, \tilde{O}, O$, and Q are constant positive definite symmetric matrices.*

This assertion is the most important part of this approach and is a direct extension of the Landau proportional+integral control to include the system given by (4). A proof of the assertion is given in Appendix A.

The object of the matrix D is to make Z(s) in (14) strictly positive real. However, as can be seen from (14), some upper and lower bounds for each of the parameters of B_p are needed to insure the positivity requirement. Under these conditions tests for positive realness such as those of Anderson [26] or Silyak [27] may prove to be useful.

In the special case when B_p is known completely, but other parameters in the plant are unknown, the following theorem of Kalman and Popov provide a satisfactory solution for D. See Landau [19].

Theorem (Kalman, Popov) [19]: The Hurwitz matrix A and matrices B,C and J define an observable and controllable system:

---

* A theorem derived by Landau [18] may be used to generalize these matrices to include positive real kernels. See Appendix A.

$$K^TK = J + J^T$$

$$\Gamma A + A^T\Gamma = -LL^T$$

$$B^T\Gamma + K^TL^T = C \tag{16}$$

In our case $Z(s) = D(SI - A_m + B_pK_m)^{-1} B_p$ with $J=0$. Therefore, if $(A_m - B_pK_m)$ is Hurwitz, then $B_p^T \Gamma = D$ will satisfy the above theorem for positive realness. Note that if $(A_m - B_pK_m)$ is not known to be Hurwitz, K_m may be set to zero as mentioned earlier and thus Γ will exist. (Recall A_m was postulated to be Hurwitz in the problem definition.)

At this point the design procedure has been completed and all that remains is to illustrate the theory by using it on a particular problem. In Chapter 4 we apply the theory to a specific nonlinear plant and simulate the resulting system.

CHAPTER 4

AN EXAMPLE

In the last chapter, a design procedure for the nonlinear M.R.A.C. problem was given in general terms. In this chapter we will apply the procedure to a specific nonlinear plant that will be forced to adapt to a linear uncoupled model.

<u>The Problem</u>:

The plant

$$\begin{aligned} \dot{w}_1 &= \frac{I_2 - I_3}{I_1} w_2 w_3 + \frac{u_1}{I_1} \\ \dot{w}_2 &= \frac{I_3 - I_1}{I_2} w_1 w_3 + \frac{u_2}{I_2} \\ \dot{w}_3 &= \frac{I_1 - I_2}{I_3} w_1 w_2 + \frac{u_3}{I_3} \end{aligned} \tag{17}$$

models the tumbling motion of an orbiting satellite whose tumbling rate far exceeds the orbiting rate (Hsu and Meyer [28]). The W's are angular velocities, I's moments of inertia, and the U's are torques due to thrust rockets. Note that $I_i > 0$ $(i = 1,2,3)$. Therefore B_p is nonsingular, and thus the perfect following conditions (B1) are satisfied.

Assume for the purposes of the example that the actual values of the parameters I_1, I_2, and I_3 are 10, 40, and 25 respectively. Thus the plant is given by

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix}_p = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 375 & 0 \\ 0 & 0 & -1.25 \end{bmatrix} \begin{bmatrix} w_2 & w_3 \\ w_1 & w_3 \\ w_1 & w_2 \end{bmatrix}_p + \begin{bmatrix} 1 & 0 & 0 \\ 0 & .025 & 0 \\ 0 & 0 & .04 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_p \tag{18}$$

Also assume the only information given about the plant is its structure (i.e., none of the parameters I_i are known exactly) and the following bounds for the R_p and B_p matrices which may have been found through experimentation. (For this plant, these bounds represent a 10:1 range over the actual values).

$$R_p \sim \begin{bmatrix} (-15,15) & 0 & 0 \\ 0 & (-15,15) & 0 \\ 0 & 0 & (-15,15) \end{bmatrix} \quad B_p \sim \begin{bmatrix} (0,1] & 0 & 0 \\ 0 & (0,1] & 0 \\ 0 & 0 & (0,1] \end{bmatrix} \tag{19}$$

Given (19), we may choose nominal values* for R_p and B_p anywhere in the given range. For convenience choose

$$\bar{R}_p = [0] \quad \text{and} \quad \bar{B}_p = [I] \tag{20}$$

Suppose the reference model is:

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix}_m = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}_m + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_m \tag{21}$$

<u>THE DESIGN</u>:

We now set the ΔK's in (B2)** to zero and substitute the nominal values:

---

* As stated in Chapter 3, choice of nominal values only affects the separation of magnitude scales.

** Since it has been shown that K_m may be set to zero, we will do so for convenience.

$$-K_p = \bar{B}_p^+ (A_m - \bar{A}_p)$$

$$K_{R_m} = \bar{B}_p^+ R_m$$

$$K_{R_p} = - \bar{B}_p^+ \bar{R}_p \qquad (22)$$

$$K_u = \bar{B}_p^+ B_m$$

Since $\bar{A}_p$ does not appear in the plant, it is set to zero. Hence

$$K_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K_{R_p} = [0] \quad , \qquad K_{R_m} = [0]$$

$$K_u = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad (23)$$

The operator D is now chosen to make Z(s) in (14) positive real. We note that for our case

$$Z(S) = D \begin{bmatrix} S+1 & 0 & 0 \\ 0 & S+1 & 0 \\ 0 & 0 & S+1 \end{bmatrix}^{-1} \begin{bmatrix} (0,1] & 0 & 0 \\ 0 & (0,1] & 0 \\ 0 & 0 & (0,1] \end{bmatrix}$$

or

$$Z(S) = D \begin{bmatrix} \frac{(0,1]}{(S+1)^2} & 0 & 0 \\ 0 & \frac{(0,1]}{(S+1)^2} & 0 \\ 0 & 0 & \frac{(0,1]}{(S+1)^2} \end{bmatrix} \qquad (24)$$

and therefore choosing D = [I] assures that Z(S) is strictly positive real.

Assign for simplicity the adaptive gains $\tilde{L} = L = \tilde{M} = M = \tilde{N} = N = \tilde{O} = O$ equal to

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \tag{25}$$

The ΔK's(0) = [0], and set S = T = P = q equal to the value in (21)

$$e_i = w_{im} - w_{ip} \quad , \quad i=1 \text{ to } 3$$

$$v_i = e_i$$

$$u_p = u_{p_1} + u_{p_2}$$

$$u_{p_{1i}} = - w_{ip} + w_{im}$$

$$\begin{aligned} u_{p_{21}} = & [\int_o^t 100\, v_1 w_{1p} d\tau + 100\, v_1 w_{1p}] w_{1p} \\ & + [\int_o^t 100\, v_1 w_{2p} w_{3p} d\tau + 100\, v_1 w_{2p} w_{3p}] w_{2p} w_{3p} \\ & + [\int_o^t 100\, v_1 u_{m_1} d\tau + 100\, v_1 u_{m_1}] u_{m_1} \end{aligned}$$

$$\begin{aligned} u_{p_{22}} = & [\int_o^t 100\, v_2 w_{2p} d\tau + 100\, v_2 w_{2p}] w_{2p} \\ & + [\int_o^t 100\, v_2 w_{1p} w_{3p} d\tau + 100\, v_2 w_{1p} w_{3p}] w_{1p} w_{3p} \\ & + [\int_o^t 100\, v_2 u_{m_2} d\tau + 100\, v_2 u_{m_2}] u_{m_2} \end{aligned}$$

$$
\begin{aligned}
u_{p_{23}} = {} & [\int_0^t 100\, v_3 w_{3p}\, d\tau + 100\, v_3 w_{3p}] w_{3p} \\
& + [\int_0^t 100\, v_3 w_{2p} w_{1p}\, d\tau + 100\, v_3 w_{2p} w_{1p}] w_{2p} w_{1p} \\
& + [\int_0^t 100\, v_3 u_3\, d\tau + 100\, v_3 u_3] u_{m_3}
\end{aligned} \tag{26}
$$

We now present the results and problems encountered with the simulation* of the adaptive system described above.

SIMULATION AND RESULTS

The main objective of the simulation is to show that adaptation worked with and without initial conditions in the plant. We were to confirm this by measuring the error vector between model and plant with and without the adaptive controller. If given the same initial conditions, the error vector of the adaptive system converged near zero and non-adaptive system didn't, then the adaptation was working.

Other objectives were to study the effects of the adaptive gain constants on error transients and convergence rates, and to study the use of plant and/or model states in the formation of the nonlinearities in the adaptive loop. The adaptive gain study is important since theoretically the gains may be arbitrarily high and thus produce arbitrarily** fast adaptation. The other study was introduced since it can be shown, by an

---

* The simulation was implemented using the C.S.M.P. language on a DEC-10 digital computer. For further information and sample simulation programs see Appendix C.

** Arbitrary up to the time optimal value.

argument similar to that used in chapter 4, that an $w_{im}w_{jm}$ integral may be used in place of, or in addition* to, the $w_{ip}w_{jp}$ integral in equation 25 and still have an asymptotically hyperstable error system.

The non-adaptive plant and reference mode described (equations 18 and 21) were both simultaneously simulated using the same sinusoidal input and zero initial conditions. An error vector was formed as the difference between the plant and reference model states, and recorded as functions of time into what now are Figures 4 to 6. These figures, along with a typical state signal of the plant (Figure 7), show that the errors are very large but bounded.

The adaptive system (equations 18, 21, and 26) was then similarly simulated with the same inputs, same initial conditions, and adaptive gains of 10 (see equation 25). The results of this simulation are given in Figures 8 and 9. Comparing Figure 4 to Figure 8 and Figure 6 to 9, it is plain to see that the error vector for the adaptive system seems to be considerably less than its non-adaptive counterpart for all time, and that it is converging to within a small neighborhood of zero.

Having shown that the system adapts for zero initial conditions, the system was then simulated with various initial conditions. Figures 10, 11 and 12 depict the state errors converging to within some neighborhood of zero for initial conditions of $w^o_{1p} = -3$, $w^o_{2p} = 1$ and $w^o_{3p} = 4$. Initial

---

* Integral has the form

$$[\int_o^t 100\ v_- w_{im} w_{jm} + 100\ v_- w_{im} w_{jm}] w_{im} w_{jm}$$

and is added on to equation 25.

conditions in the same order of magnitude produced similar results, however when the initial conditions were made an order of magnitude higher it yielded inconclusive results (more on this last point later in this chapter).

Figures 13, 14, and 15 are the same error plots of the Figures 10, 11, and 12 with the first second deleted and the same scale blown up. Likewise for Figures 16, 17, and 18, except the adaptive gains were increased to $10\sqrt{3} \sim 17$. Comparison among these figures show that the higher gains produced faster initial decreases in the state errors by a factor of at least 1.5 to 2. However, the rates leveled off after about 10 seconds to produce little improvement, specially for w_{3p}. This type of effect has been observed earlier while simulating an adaptive observer.* It appears that high gains tend to increase the initial improvements but induce damped tail oscillations as the error approaches zero. Quite often, decreasing the adaptive gains in the linear terms will increase the damping rate at the tail end at the expense of initial rate decreases.

The effects of replacing by, or the adding of an $w_{im}w_{jm}$ term were tested by similar simulations as above. Given the same sinusoidal inputs, adaptive gains of 10, and initial conditions ($w_{1p}^o = 3$, $w_{2p}^o = -1$, and $w_{3p}^o = 0$); systems using the $w_{im}w_{jm}$ functional, using the $w_{ip}w_{jm}$ functional, and using both together were simulated. The results are shown in Figures 19 to 23. Comparing Figures 19, 21 with 23, and 20 with 22 we see, at least for this system, that no distinct advantages exist for any of the cases. That is, no convergence properties establish one case better than the other, but this does not mean that a noisy version of the system will exhibit the same properties.

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*This effect was noticed while simulating an adaptive observer using the Narendra structure [30].

System signals without adaptation and with zero initial conditions.

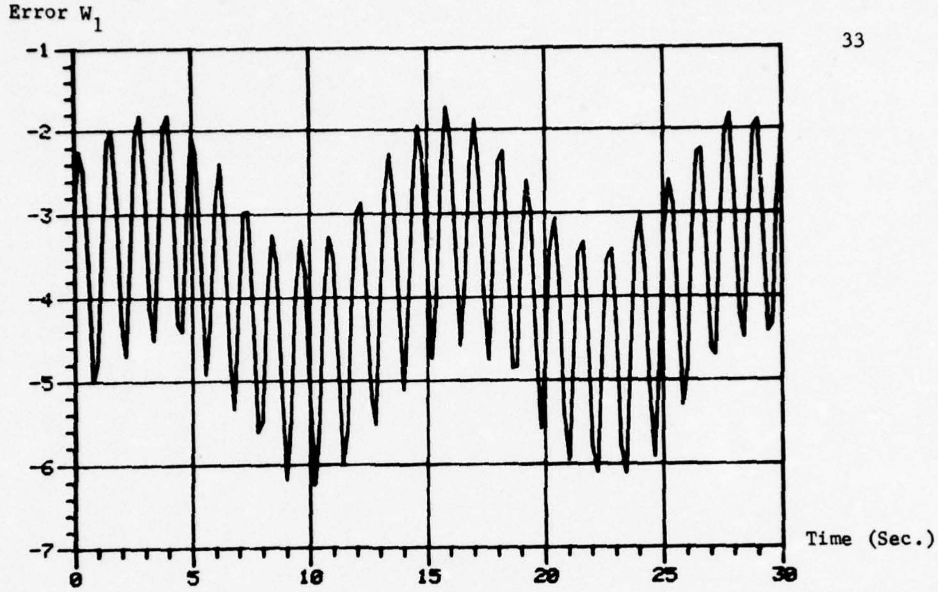

Figure 4. Error in w_1 state in nonadaptive system.

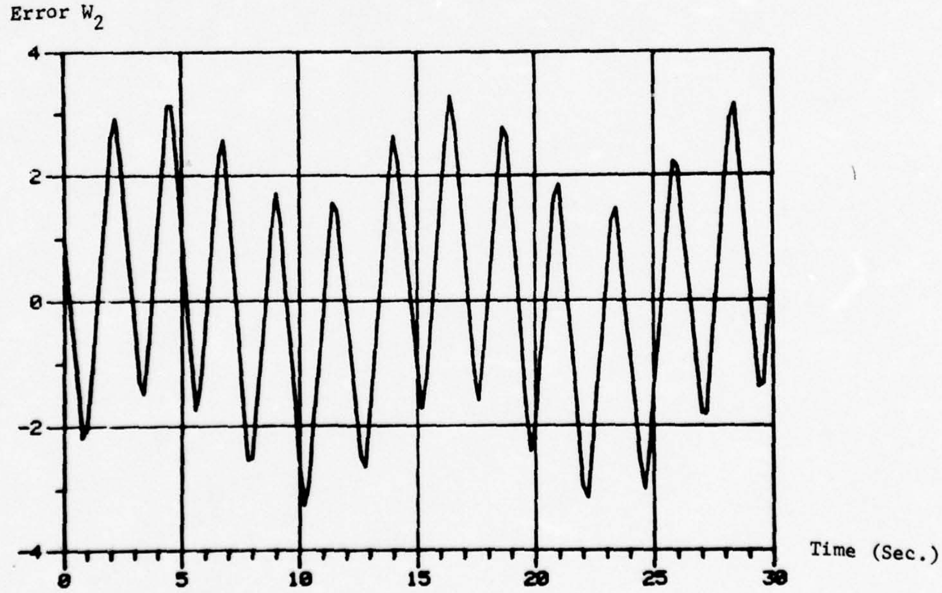

Figure 5. Error in w_2 state in nonadaptive system.

Initial Conditions: $w^o_{1p} = 0$, $w^o_{2p} = 0$, $w^o_{3p} = 0$

Integration Interval = .02 sec.

Print Interval = .2 sec.

Inputs: $u_{m1} = u_{m2} = u_{m3} = \sin .5t$

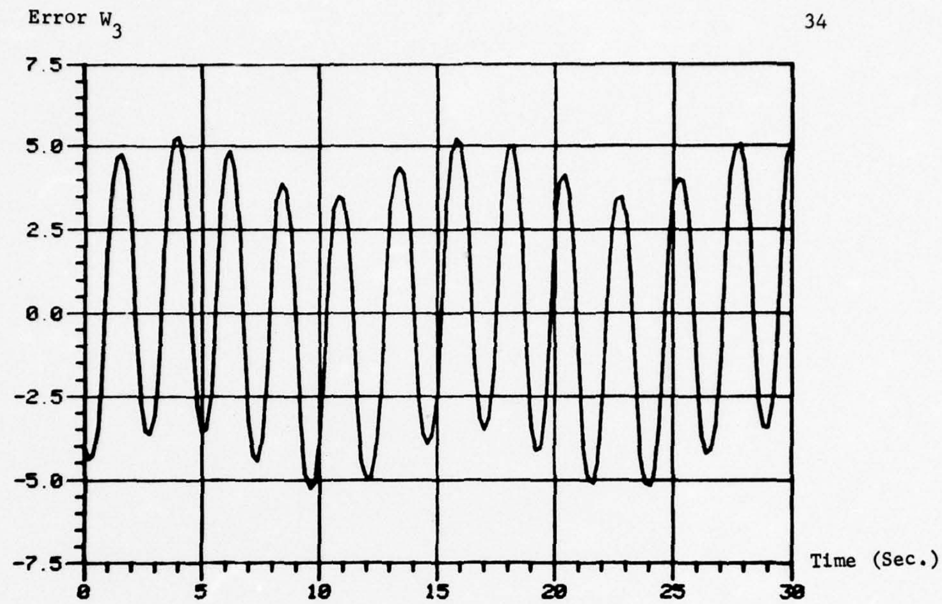

Figure 6. Error in w_3 state in nonadaptive system.

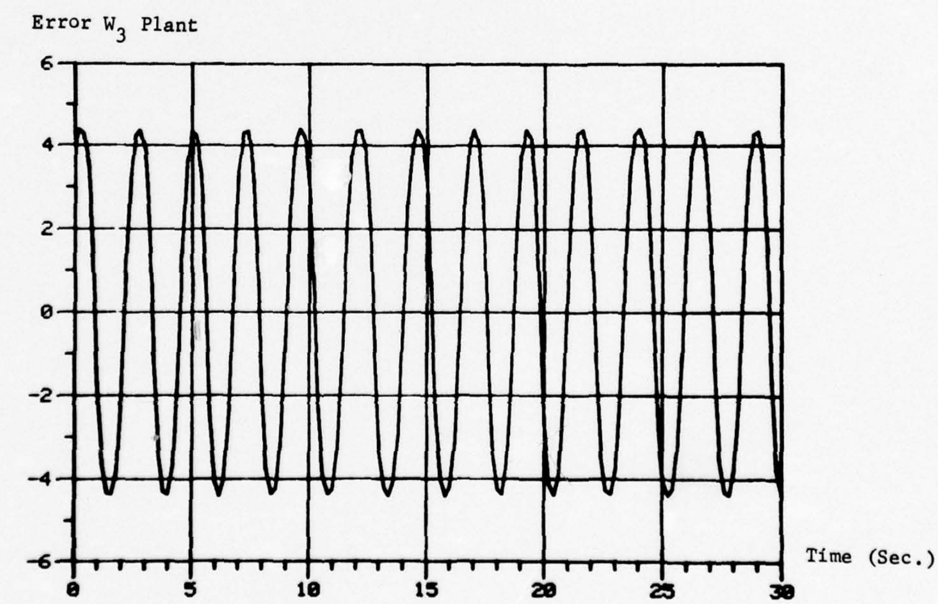

Figure 7. Typical plant state signal in nonadaptive system.

System signals with adaptation and with zero initial conditions.

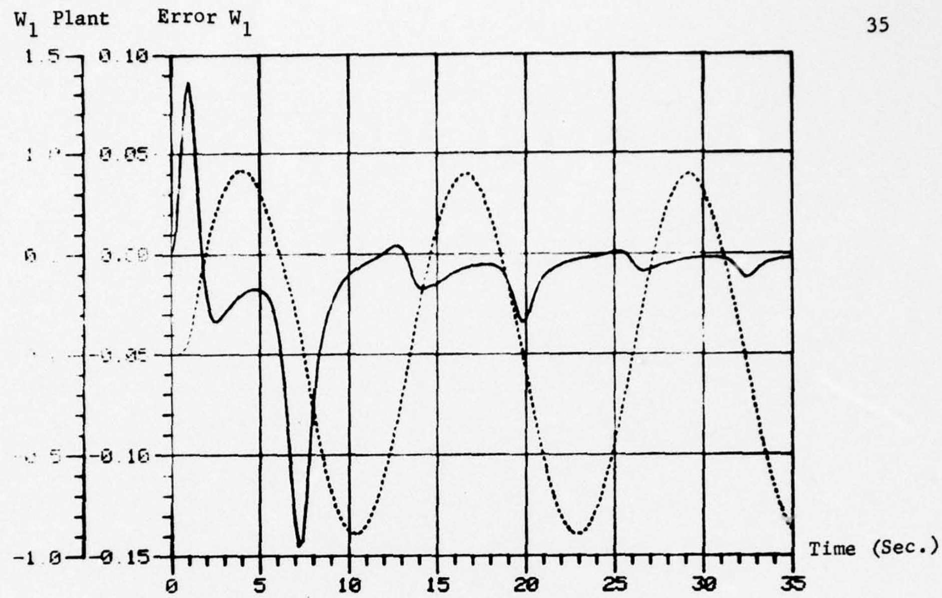

Figure 8. w_1 plant state and corresponding performance error for adaptive system.

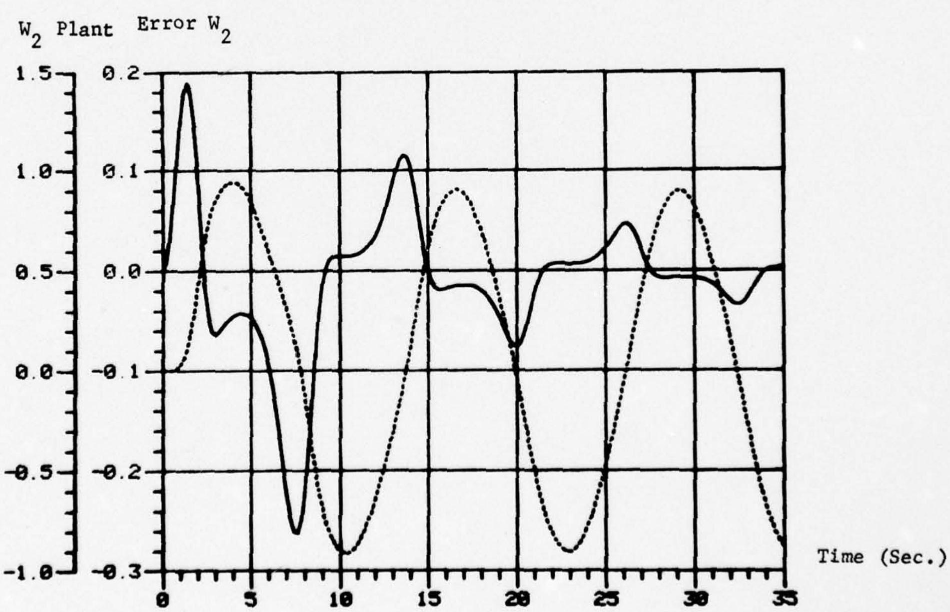

Figure 9. w_2 plant state and corresponding performance error for adaptive system.

Initial Conditions: $w^o_{1p} = 3$, $w^o_{2p} = -1$

$w^o_{3p} = 4$

Integration Interval = .005 sec.

Printing Interval = .2 sec.

Inputs: $u_{m1} = u_{m2} = u_{m3} = \sin .5t$

Adaptive Gains = 10

System signals with adaptation and initial conditions.

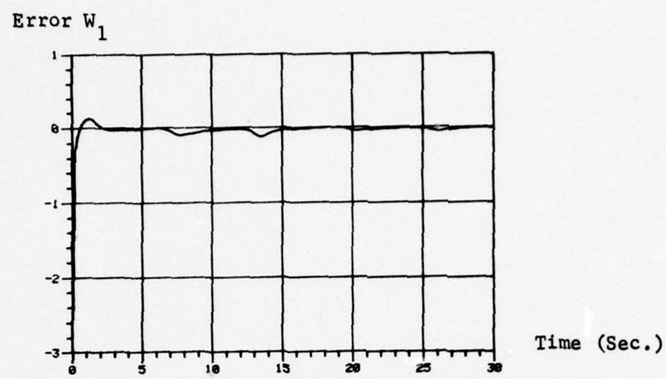

Figure 10. w_1 performance error.

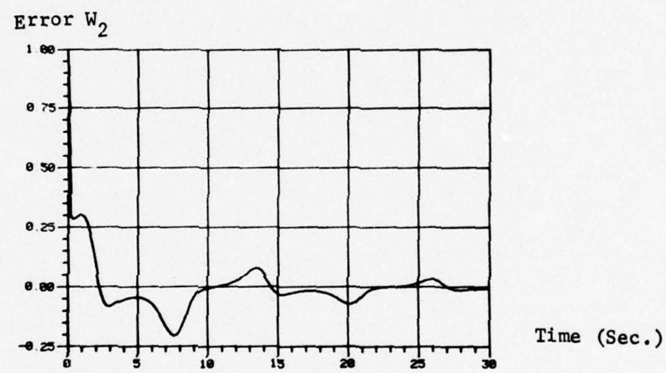

Figure 11. w_2 performance error.

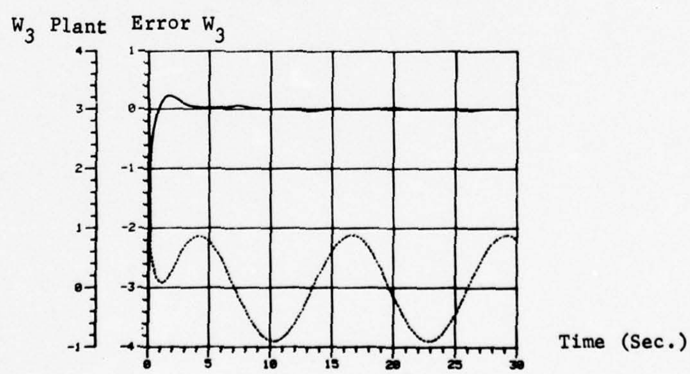

Figure 12. w_3 plant state output and corresponding performance error.

Initial Conditions: $w_{1p}^{o} = 3$, $w_{2p}^{o} = -1$

$w_{3p}^{o} = 4$

Integration Interval = .005 sec.

Printing Interval = .2 sec.

Inputs: $u_{m1} = u_{m2} = u_{m3} = \sin .5t$

Adaptive Gains = 10

System signals with adaptation, initial conditions, and adaptive gains equal 10. First second of response is deleted to provide a more detailed view of transient.

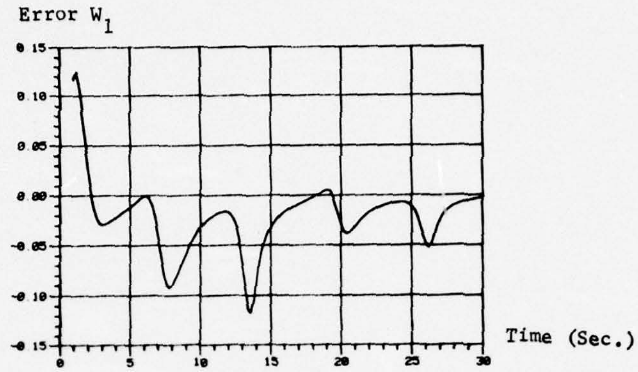

Figure 13. Scaled w_1 performance error.

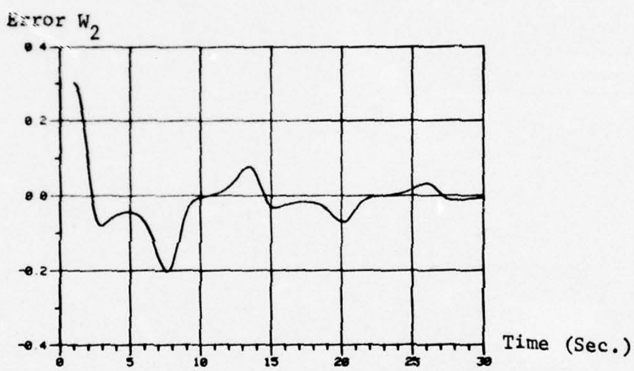

Figure 14. Scaled w_2 performance error.

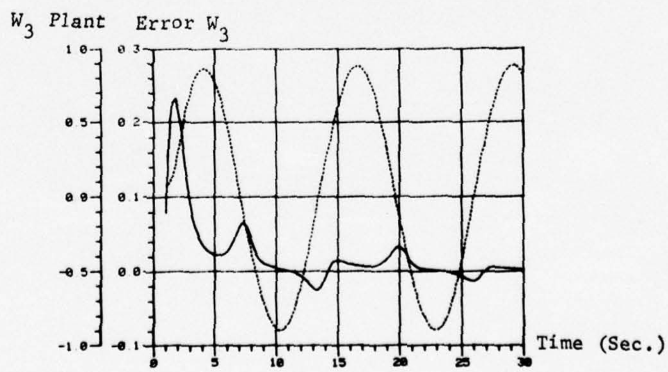

Figure 15. Scaled w_3 plant state output and corresponding performance error.

Initial Conditions: $w^o_{1p} = 3$, $w^o_{2p} = -1$, $w^o_{3p} = 4$

Integration Interval = .002 sec.

Printing Interval = .15 sec.

Inputs: $u_{m1} = u_{m2} = u_{m3} = \sin .5t$

Adaptive Gains = 17

System signals with adaptation, initial conditions, and adaptive gains equal 17. First second of response is deleted for more detailed view of transient. Compare with Figures 13, 14, and 15.

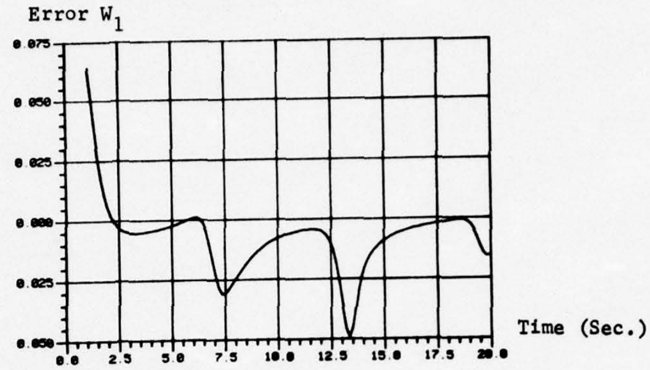

Figure 16. Scaled w_1 performance error with increased adaptive gains.

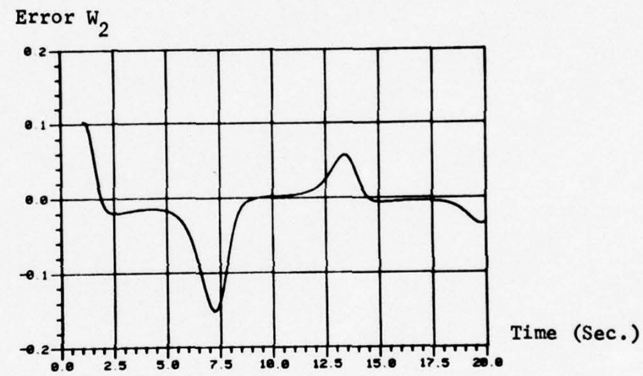

Figure 17. Scaled w_2 performance error with increased adaptive gains.

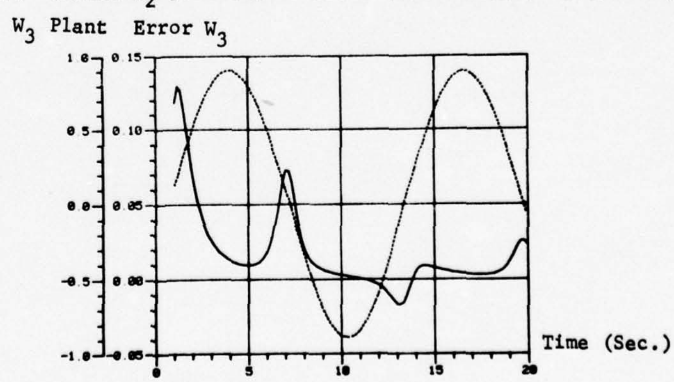

Figure 18. Scaled w_3 plant state and performance error with increased adaptive gains.

Initial Conditions: $w^o_{1p} = 3$, $w^o_{2p} = -1$

$w^o_{3p} = 0$

Integration Interval = .02 sec.

Printing Interval = .2 sec.

Inputs: $u_{m1} = u_{m2} = u_{m3} = \sin .5t$

Adaptive Gains = 10

System signals with w_p's functional adaptation only.

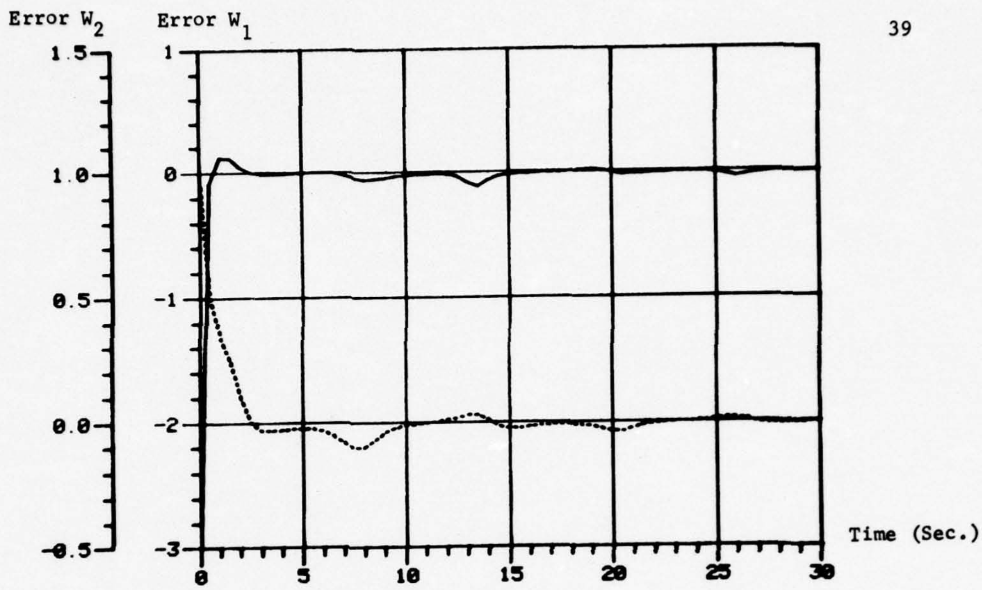

Figure 19. w_1 and w_2 performance errors with w_p's functional adaptation only.

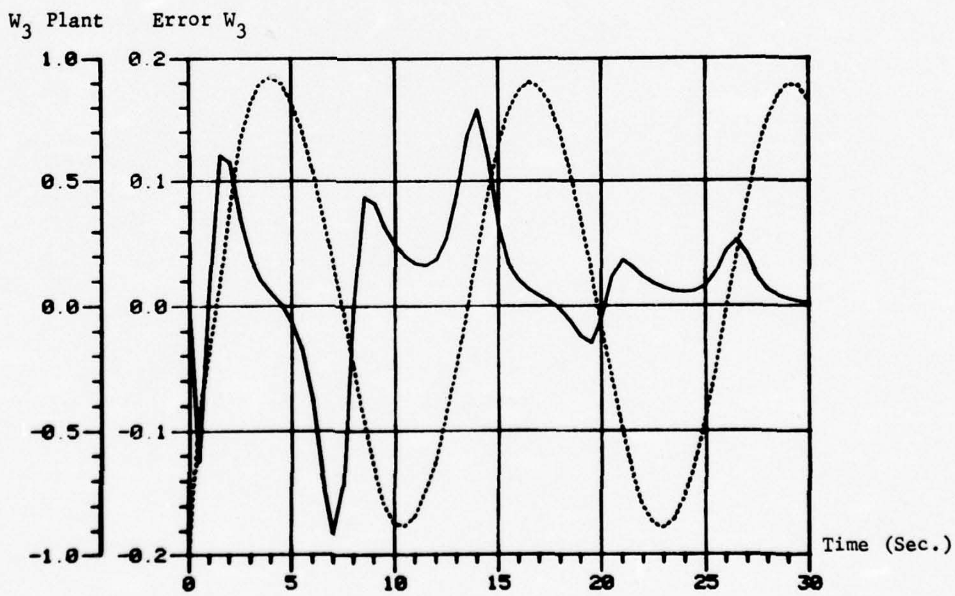

Figure 20. w_3 plant state and corresponding performance error with w_p's functional adaptation only.

Initial Conditions: $w_{1p}^o = 3$, $w_{2p}^o = -1$

$w_{2p}^o = 0$

Integration Interval = .02 sec.

Printing Interval = .2 sec.

Inputs: $u_{m1} = u_{m2} = u_{m3} = \sin .5t$

Adaptive Gains = 10

System Signals with w_m's functional adaptation only.

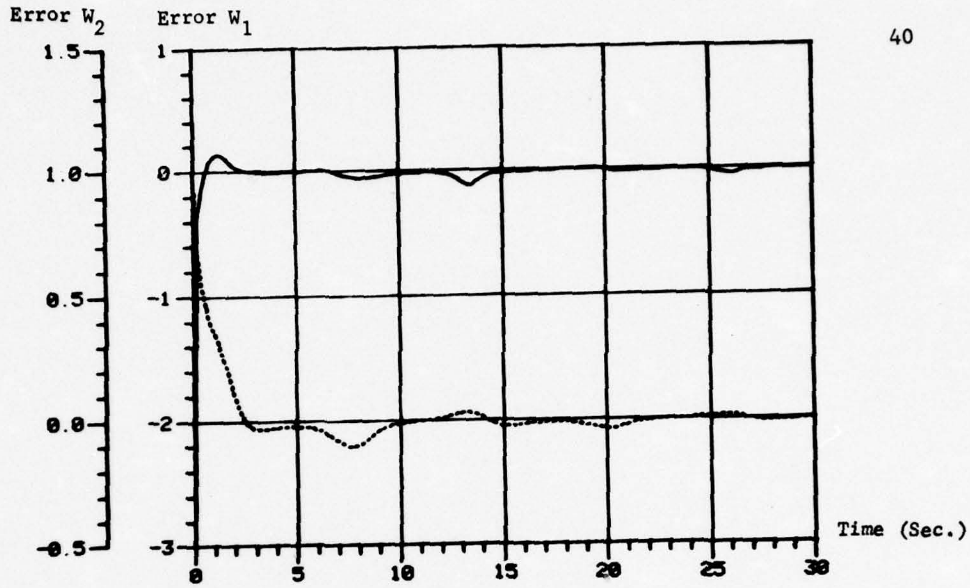

Figure 21. w_1 and w_2 performance errors with w_m's functional adaptation only.

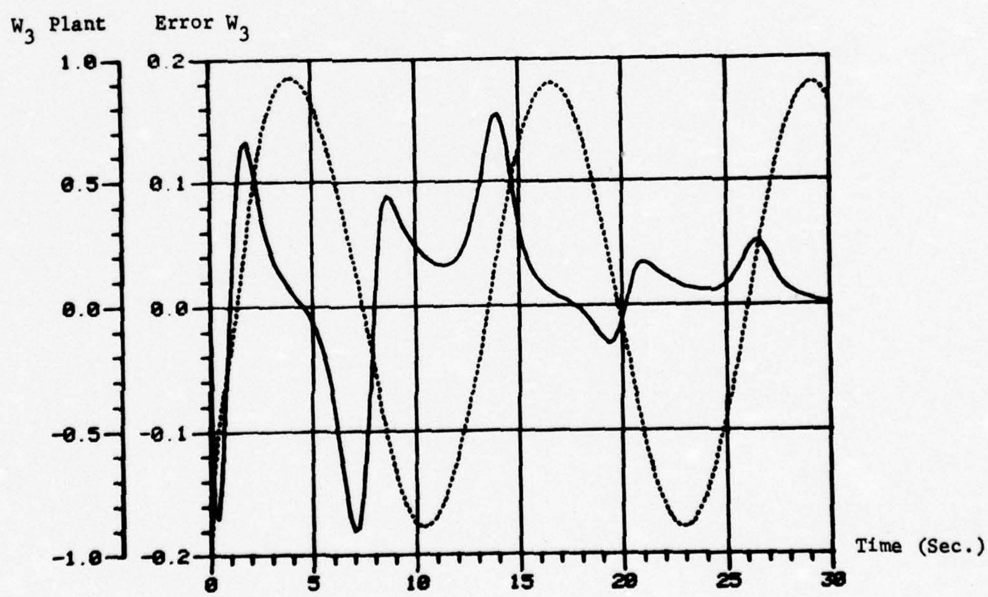

Figure 22. w_3 plant state and corresponding performance error with w_m's functional adaptation only.

Initial Conditions: $w_{1p}^{o} = 3$, $w_{2p}^{o} = -1$

$w_{2p}^{o} = 0$

Integration Interval = .02 sec.

Printing Interval = .2 sec.

Inputs: $u_{m1} = u_{m2} = u_{m3} = \sin .5t$

Adaptive Gains = 10

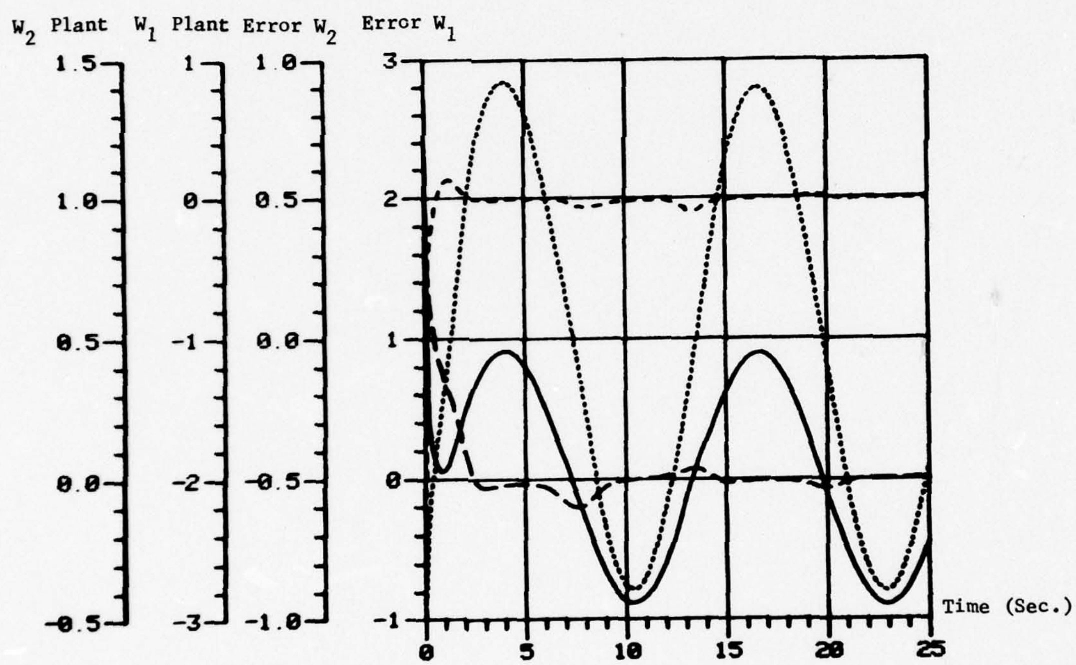

Figure 23. System signals with w_m's and w_p's functional adaptation.

Problems Encountered

Several problems appeared during the simulation phase that hindered the evaluation of several of the performance checks. It appears that the C.S.M.P. language we used was implemented with a single precision second order Runge Kutta integration routine, and it was introducing truncation and convergence problems. For example, while simulating for initial conditions of $w_{p_1}^{o} = 40$, $w_{p_2}^{o} = -30$ and $w_{p_3}^{o} = 15$, the routine overflowed with an integration interval of .000025 sec., underflowed for a .000020 sec. value, and converged for a .000021 value. Clearly inconclusive results since it is impossible to tell whether the system adapted or whether the integration routine truncated system values to force it to converge. A related problem was the number of iterations needed to carry out the simulation assuming everything worked, the integration interval had to be .005 seconds for many cases. This implies that a 20 second run would need 4000 iterations or about 1½ minutes of CPU time, clearly very expensive for a single simulation. Similar problems to all the above were also encountered while increasing the adaptive gains.

CHAPTER 5

NEW DIRECTIONS AND CONCLUSION

PROBLEMS UNSOLVED:

To make the design procedure in Chapter 4 competitive, several problems must first be solved. In particular, a systematic method of generating the linear matrix D would not only simplify calculations but it would extend the method to include unknown B_p matrix parameters. A clue to such a result probably lies in works dealing with the positive real functions criterion [26,27]. Another very important problem is the generation of state variable measurements without using derivative operations. Though such a method has been found for linear systems, the nonlinear case is still open.

A problem of practical nature is that of formulating a systematic way of choosing the adaptive gains of the controller to meet performance criteria. Such a result would be very useful in designing controllers for systems with moderately fast time-varying parameters or with infrequent* discontinuous parameter perturbations.

Related problems of an algebraic nature are the formulation of necessary and sufficient conditions for perfect model following in the nonlinear case, and the existence of a minimal parameter set structure to uniquely define nonlinear systems.** These solutions would generate a minimal number of adaptive loops and guarantee perfect adaptation.

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* With respect to adaptation time transients.

** As it stands, there is an adaptive loop for every parameter in the nonlinearity which must be adapted plus minimal number for the A_p and B_p matrices. Thus, possibly too many to be practical.

If the minimal parameter problem described above is solved, the following approach may prove useful in the structural adaptation problem when the order of the system is known.

Consider the class of nonlinear functions having factorable unknown parameters and is also analytic in the region of question. Then, if the F_{P_α} matrix of (4) is assumed to be composed of the first few terms of the nonconstant part of the Taylor expansion terms, the ΔK_{R_P} may be forced to adapt (by the method shown in Chapter 4) to the coefficients of the Taylor expansions of the unknown functions. Since the functions were assumed to be analytic, it would seem that the model following error should at least be bounded by a function involving the order of the error in the Taylor expansion. Thus, the possibility of structural adaptation may exist by a slight extension of the work in Chapter 4 and a minimal parameter structure.

CONCLUSION:

Parameter adaptive control has been studied as alternate approach when conventional controls fail. In particular, the M.R.A.C. structure was discussed in conjunction with stability synthesis. Lyapunov and hyperstability were then presented as general but mathematically complex synthesis methods. Choosing hyperstability, the theory was successfully applied. Results show that the approaches of Landau [19] and Erzberger [29] may be combined to yield a new hyperstable M.R.A.C. for a class of nonlinear systems.

In solving the problem, two useful assertions were introduced, one providing sufficient conditions for perfect model following and the other an adaptive algorithm for the nonlinear plant. A design example was then formulated and the resulting system simulated. The results confirmed the effectiveness of the approach.

APPENDIX A

DERIVATION OF ADAPTIVE LAWS

The following assertion provides an algorithm for generating the adaptive ΔK's.

Assertion: The following choices for the ΔK's satisfy (13).

$$\Delta K_{R_m} = \int_0^t \tilde{L}V[Sf_m]^T d\tau + LV[Sf_m]^T + \Delta K_{R_m}(0)$$

$$\Delta K_{R_p} = \int_0^t \tilde{O}V[Qf_p]^T d\tau + OV[Qf_p]^T + \Delta K_{R_p}(0)$$

$$\Delta K_p = \int_0^t \tilde{N}V[Px_p]^T d\tau + NV[Px_p]^T + \Delta K_p(0)$$

$$\Delta K_u = \int_0^t \tilde{M}V[Tu_m]^T d\tau + MV[Tu_m]^T + \Delta K_u(0)$$

where $\tilde{L}$, L, S, $\tilde{M}$, M, T, $\tilde{N}$, N, P, $\tilde{O}$, O, and Q are constant positive definite symmetric matrices.

Proof: See also Appendix II of [19].

From (9) and (13)

$$\begin{aligned}\eta(0,t_1) = &\int_0^{t_1} N^T[K_{R_m} + \Delta K_{R_m} - B_p^+ R_m]F_m \, d\tau \\ &+ \int_0^{t_1} N^T[K_u + \Delta K_u - B_p^+ B_m]u_m \, d\tau \\ &+ \int_0^{t_1} N^T[K_m + \Delta K_p - K_p - B_p^+ (A_m - A_p)]x_p \, d\tau \\ &+ \int_0^{t_1} N^T[B_p^+ R_p + \Delta K_{R_p} + K_{R_p}]F_p \, d\tau \end{aligned} \qquad (A1)$$

Substituting (15) into (lc)

$$\begin{aligned}
\eta(0,t_1) = &\int_0^{t_1} N^T\{\int_0^{t} \tilde{L}N[Sf_m]^T\, d\tau' - A_o\}F_m\, d\tau \\
&+ \int_0^{t_1} N^T\{\int_0^{t} \tilde{M}V[Tu_m]^T\, d\tau' - B_o\}u_m\, d\tau \\
&+ \int_0^{t_1} N^T\{\int_0^{t} \tilde{N}V[Px_p]^T\, d\tau' - C_o\}x_p\, d\tau \\
&+ \int_0^{t_1} N^T\{\int_0^{t} \tilde{O}V[Qf_p]^T\, d\tau' - D_o\}F_p\, d\tau \\
&+ \int_0^{t_1} (N^T LV)[F_m^T S F_m] d\tau \\
&+ \int_0^{t_1} (N^T MV)[u_m^T T u_m] d\tau \\
&+ \int_0^{t_1} (N^T NV)[x_p^T P x_p] d\tau \\
&+ \int_0^{t_1} (N^T OV)[F_p^T Q F_p] d\tau .
\end{aligned} \qquad {}^*(A2)$$

where

$$A_o = B_p^+ R_m - K_{R_m} - \Delta K_{R_m}(0) = \text{constant matrix}$$

$$B_o = B_p^+ B_m - K_u - \Delta K_u(0) = \text{constant matrix}$$

$$C_o = B_p^+(A_m - A_p) + K_p - K_m - \Delta K_p(0) = \text{constant matrix}$$

$$D_o = -B_p^+ R_p - K_{R_p} - \Delta K_{R_p}(0) = \text{constant matrix.}$$

---

*In the case where $\beta > 1$ and $\rho > 1$ we would have similar terms to these.

Sufficient conditions that $\eta(0,t_1) \geq -\gamma_o^2 - \gamma_o$ are that each term of (2c) be greater than $-\gamma_{oi}^2$, where i ranges over the number of terms. Thus the last four terms of (2c) satisfy this condition since L, S, M, T, N. P, O, and Q are all positive definite. Hence, it remains to be shown that the terms of the type

$$\int_o^{t_1} V^T\{\int_o^t \tilde{\xi} V[\Omega f(\bar{x} \text{ XOR } \bar{u})]^T d\tau' - \psi_o\} f(\bar{x} \text{ XOR } \bar{u}) d\tau \qquad (A3)$$

are greater than $\geq -\gamma_{oi}^2$. Where $\bar{\xi}$ and Ω represent the symmetric positive definite matrices, $f(\bar{x}\,(\text{XOR})\,\bar{u})$ represents some function involving (x_p OR x_m) XOR (u_m OR u_p), and ψ_o represents the constants.

Now, note that symmetric positive definite matrices may be decomposed into two parts; i.e. for Ω positive definite symmetric matrix

$$\Omega = \Omega_1^T \, \Omega_1.$$

Therefore if we define $\tilde{\xi} = \tilde{\xi}_1^T \, \tilde{\xi}_1$; $\Omega = \Omega_1^T \, \Omega_1$; $\tilde{V} = \tilde{\xi}_1 V$; $\tilde{f} = \Omega_1 f$; $\tilde{\psi}_o = \tilde{\psi}_o(\Omega_1)^{-1}$; then (3c) becomes

$$\int_o^{t_1} \tilde{V}^T[\int_o^t \tilde{V}\tilde{f}^T d\tau' - \tilde{\psi}_o]\tilde{f}\, d\tau = \sum_i \sum_J \int_o^{t_1} \tilde{V}_i \tilde{f}_J [\int_o^t V_i \tilde{f}_J d\tau' - \tilde{\psi}_{o_{iJ}}] d\tau. \qquad (A4)$$

But

$$\int_o^t \gamma(\tau)[\int_o^\tau \gamma(\tau')d\tau']d\tau = \frac{1}{2}[\int_o^t \gamma(\tau)d\tau]^2 \qquad (A5)$$

since if $u = \int_o^\tau \gamma(\tau')d\tau'$ and $dv = \gamma(\tau)d\tau$ then

$$\int_o^t \gamma(\tau)[\int_o^\tau \gamma(\tau')d\tau']d\tau = uv| - \int vdu = [\int_o^\tau \gamma(\tau)d\tau]^2 - \int_o^\tau \gamma(\tau)[\int_o^\tau \gamma(\tau')d\tau']d\tau$$

and therefore

$$\int_0^t \gamma(\tau)[\int_0^\tau \gamma(\tau')d\tau']d\tau = \frac{1}{2}[\int_0^t \gamma(\tau)d\tau]^2 \quad .$$

Hence (4c) equals

$$\frac{1}{2}\sum_i\sum_j\{[\int_0^{t_1}\tilde{V}_i\tilde{f}_j\,d\tau - \psi^2_{oij}]^2 - \psi^2_{oij}\}$$

which is obviously

$$\geq -\frac{1}{2}\sum_i\sum_j(\psi_{oij})^2$$

It is now possible to show that the positive definite matrices in (15) may be generalized to the class of positive definite kernels [18].

<u>Definition</u>: The square matrix kernel K(t) is termed "positive definite" if for each interval $[T_o,T]$ and all vector functions f(t) piecewise continuous on $[T_o,T]$, the following inequality holds:

$$\eta(T_o,T) = \int_{T_o}^{T} f'(t)[\int_{T_o}^{T} K(t-\tau)f(\tau)d\tau]dt \geq 0 \tag{A6}$$

<u>Lemma [18]</u>: For the class of kernels K(t) for which the Laplace transform exists, the necessary and sufficient conditions, in order that K(t) be a positive definite kernel, is that its Laplace transform be a positive real transfer matrix.

Therefore, by the above, $\tilde{L},\tilde{M},\tilde{N}$, and $\tilde{O}$ may be generalized to be positive real matrix operators instead of constants.

APPENDIX B

The following assertion provides sufficient conditions under which control u_p will achieve perfect model following.

Assertion: Sufficient conditions for the control (6) to force the plant (4) to follow the model (5) are:

1. $(I - B_p B_p^+)(A_m - A_p) = 0$

   $(I - B_p B_p^+) B_m = 0$

   $(I - B_p B_p^+) R_m = 0 \; ; \quad (I - B_p B_p^+) R_p = 0 \; ; \quad \gamma = 1 \text{ to } \rho, \quad \alpha = 1 \text{ to } \beta$

$$\tag{B1}$$

2. $\lim_{t \to \infty} e \to 0$

where $\quad e \triangleq X_m - X_p \quad$ and $\quad B_p^+ \triangleq (B_p^T B_p)^{-1} B_p^T$

Furthermore a set of controls that will achieve perfect following under the above conditions are characterized by:

$$K_m + \Delta K_p - K_p = B_p^+(A_m - A_1)$$

$$K_u + \Delta K_u = B_p^+ B_m$$

$$K_{R_p} + \Delta K_{R_p} = -B_p^+ R_p \; ; \quad K_{R_m} + \Delta K_{R_m} = B_p^+ R_m \; ; \tag{B2}$$

<u>Proof</u>:* Note that perfect model following implies e=0 and $\dot{e}$=0, and conversely. Therefore we will assume e=0 and then show under what conditions $\dot{e}$ will equal zero. To this end we define

$$e \stackrel{\Delta}{=} X_m - X_p$$

and subtract (4) from (5) with the control (3) with ρ=1 and β=1

$$\dot{e} = A_m X_m + R_m F_m + B_m u_m - A_p X_p - R_p F_p - B_p[(\Delta K_p - K_p)X_p + (\Delta K_{R_p} + K_{R_p})F_p + (K_{R_m} + \Delta K_{R_m})F_m + (K_u + \Delta K_u)u_m] \tag{B3}$$

requiring e=0 and $\dot{e}$=0 implies $x_m = x_p$, therefore (B3) becomes

$$\dot{e} = 0 = [A_m - A_p - B_p(K_m + \Delta K_p - K_p)]X_p + [R_m - B_p(K_{R_m} + \Delta K_{R_m})]F_m + [- R_p - B_p(K_{R_p} + \Delta K_{R_p})]F_p + [B_m - B_p(K_u + \Delta K_u)]u_m \tag{B4}$$

but sufficient conditions for (B4) to hold are that each term equal zero for all x_p, F_m, F_p and u_m, therefore

$$A_m - A_p - B_p(K_m + \Delta K_p - K_p) = 0 \; ; \quad R_m - B_p(K_{R_m} + \Delta K_{R_m}) = 0 \; ;$$
$$- R_p - B_p(K_{R_p} + \Delta K_{R_p}) = 0 \; ; \quad B_m - B_p(K_u + \Delta K_u) = 0 \; ; \tag{B5}$$

define a set of perfect following controls. Solving for K's

$$K_m + \Delta K_p - K_p = B_p^+(A_m - A_p) \; ; \quad K_{R_m} + \Delta K_{R_m} = B_p^+ R_m \; ;$$
$$K_{R_p} + \Delta K_{R_p} = -B_p^+ R_p \; ; \quad K_u + \Delta K_u = B_p^+ B_m \tag{B6}$$

---

* Uses the techniques of [29] and [25].

Note however that to justify the use of the pseudo-inverse, we have to formulate the conditions under which (B6) satisfy (B5) [29]. Therefore substituting (B6) into (B5) we get

$$(I - B_pB_p^+)(A_m - A_p) = 0 \ ; \qquad (I - B_pB_p^+)\, R_m = 0 \ ;$$
$$(I - B_pB_p^+)\, R_p = 0 \qquad ; \qquad (I - B_pB_p^+)\, B_m = 0 \ ; \tag{B7}$$

Note also that when $\beta > 1$ and/or $\rho > 1$ we get similar terms to the above because the functions distribute, i.e.

$$(I - B_pB_p)^+\, R_{p_\alpha} = 0 \ ; \qquad (I - B_pB_p^+) R_{m_\gamma} = 0 \ ;$$
$$K_{R_{m_\gamma}} + \Delta K_{R_{m_\gamma}} = B_p^+\, R_{m_\gamma} \ ; \qquad K_{R_{p_\alpha}} + \Delta K_{R_{p_\alpha}} = \beta_p^+\, R_{p_\alpha} \ ;$$
$$\alpha = 1,\ldots,\beta \quad \text{and} \quad \gamma = 1,\ldots,\rho$$

Thus we have shown that when (B7) is satisfied, (B6) may be used to make $\dot{e}=0$. Therefore, since e is assumed to equal zero and $\dot{e}=0$, the assertion is established.

APPENDIX C

SIMULATION PROGRAMS

All the simulations of Chapter 6 were made on a DEC-10 digital computer using the C.S.M.P. simulation program. This program is quite easy to use since it treats the elements of a system block diagram as a topological network. This means that each statement is given a function type (integration, summer, multiplier, etc.) and the statement numbers of the operands. Initial conditions and parameters for the functions are given after all function statements have been declared. These are then coded by reusing the statement number of the particular block and then specifying parameter values to the inputs of the blocks by a one-to-one correspondence.

The two programs in this section are typical of all the simulation programs made. NLMRØ.DAT, for example, was used for the simulation of the error signals of the nonadaptive system. NLMRF.DAT was used for simulating the adaptive system with initial conditions and adaptive gains of 10.

Table 2. Program NLMRØ.DAT Simulates Nonadaptive System

```
 1   I    70     4    74   $WP1
 2   I    70     5    74   $WP2
 3   I    70     6    74   $WP3
 4   X     2     3         $WP2 * WP3
 5   X     1     3         $WP1 * WP3
 6   X     1     2         $WP1 * WP2
 7   I    70     7    74   $WM1
 8   I    70     8    74   $WM2
 9   I    70     9    74   $WM3
13   +     7    -1         $E1 = V1
14   +     8    -2         $E2 = V3
15   +     9    -3         $E3 = V3
70   K                     $K=0
71   K                     $K=1
72   K                     $K=?
73   X    76    72         $X*TIME
74   I    73    71         $SIN(X*TIME)
0
 1   3   1.5    .1
 2  -1   .375  .025
 3   4  -1.25  .04
 7   0   -1     1
 8   0   -1     1
 9   0   -1     1
70   0
71   1
72  .5
```

Table 3. Program NLMRF.DAT Simulates Adaptive System with Initial Conditions

1	I	70	4	67	$WP1
2	I	70	5	68	$WP2
3	I	70	6	69	$WP3
4	X	2	3		$WP2 * WP3
5	X	1	3		$WP1 * WP3
6	X	1	2		$WP1 * WP2
7	I	70	7	74	$WM1
8	I	70	8	74	$WM2
9	I	70	9	74	$WM3
10	X	8	9		$WM2 * WM3
11	X	7	9		$WM1 * WM3
12	X	7	8		$WM1 * WM2
13	+	7	-1		$E1 = V1
14	+	8	-2		$E2 = V3
15	+	9	-3		$E3 = V3
16	X	13	1		$V1 * WP1
17	X	14	2		$V2 * WP2
18	X	15	3		$V3 * WP3
19	X	13	4		$V1 * WP2 * WP3
20	X	14	5		$V2 * WP1 * WP3
21	X	15	6		$V3 * WP1 * WP2
22	X	13	10		$V1 * WM2 * WM3
23	X	14	11		$V2 * WM1 * WM3
24	X	15	12		$V3 * WM1 * WM2
25	X	13	74		$V1 * UM1
26	X	14	74		$V2 * UM2
27	X	15	74		$V3 * UM3
28	I	70	16		$INTEGRATION OF UP1 TERMS
29	I	70	19		
30.	I	70	22		
31	I	70	25		
32	I	70	17		$INTEGRATION OF UP2 TERMS
33	I	70	20		
34	I	70	23		
35	I	70	26		
36	I	70	18		$INTEGRATION OF UP3 TERMS
37	I	70	21		
38	I	70	24		
39	I	70	27		
40	W	28	16	71	$SUMMING UP1 TERMS
41	W	29	19		
42	W	30	22		
43	W	31	25	71	
44	W	32	17	71	$SUMMING UP2 TERMS
45	W	33	20		
46	W	34	23		
47	W	35	26	71	
48	W	36	18	71	$SUMMING UP3 TERMS
49	W	37	21		
50	W	38	24		
51	W	39	27	71	
52	X	40	1		$FORMING PRODUCTS OF UP1
53	X	41	4		
54	X	42	10		
55	X	43	74		
56	X	44	2		$FORMING PRODUCTS OF UP2
57	X	45	5		
58	X	46	11		
59	X	47	74		
60	X	48	3		$FORMING PRODUCTS OF UP3

```
61    X     49      6
62    X     50     12
63    X     51     74
64    +     52     53      $SEMISUM UP1
65    +     56     57      $SEMISUM UP2
66    +     60     61      $SEMISUM UP3
67    +     64     55      $UP1
68    +     65     59      $UP2
69    +     66     63      $UP3
70    K                    $K=0
71    K                    $K=1
72    K                    $K=?
73    X     76     72      $X*TIME
74    I     73     71      $SIN(X*TIME)
0
 1    3     1.5     .1
 2   -1     .375   .025
 3    4   -1.25    .04
 7    0     -1      1
 8    0     -1      1
 9    0     -1      1
28    0    100
29    0    100
30    0    100
31    0    100
32    0    100
33    0    100
34    0    100
35    0    100
36    0    100
37    0    100
38    0    100
39    0    100
40    1    100     -1
41    1    100
42    1    100
43    1    100      1
44    1    100     -1
45    1    100
46    1    100
47    1    100      1
48    1    100     -1
49    1    100
50    1    100
51    1    100      1
70    0
71    1
72   .5
```

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